Estimates for polynomial systems defining irreducible smooth complete intersections

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General setting

For $f_1, \ldots, f_s \in K[X_0, \ldots, X_n]$ $(s < n)$ homogeneous of degrees $\boldsymbol{d} := (d_1, \ldots, d_s)$, consider the projective variety

$$V := V(f_1, \ldots, f_s) := \{\boldsymbol{x} \in \mathbb{P}^n_{\overline{K}} : f_1(\boldsymbol{x}) = 0, \ldots, f_s(\boldsymbol{x}) = 0\}.$$

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Question: what can be said on V for a **general** $f_1, \ldots, f_s$?

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$$P_t(\Lambda_\alpha : \alpha \in \mathbb{N}_d^{n+1}) \in K[\Lambda_\alpha : \alpha \in \mathbb{N}_d^{n+1}]$$

such that $f = \sum_{\alpha \in \mathbb{N}_d^{n+1}} \lambda_\alpha \boldsymbol{X}^\alpha$ is not absolutely irreducible $\Leftrightarrow$

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of deg $(n+1)(d-1)^n$ s.t. $f = \sum_{\alpha \in \mathbb{N}_d^{n+1}} \lambda_\alpha \boldsymbol{X}^\alpha$ is not smooth $\Leftrightarrow$

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[Gathen-Viola-Ziegler, 2013]: $\left| \dfrac{A_{n,d}(\mathbb{F}_q)}{P_{n,d}(\mathbb{F}_q)} - 1 \right| \leq 8q^{-\binom{n+d-1}{n-1}+n}.$

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On the other hand, from the Boole bound we obtain:

$$\left|\frac{N_{n,d}(\mathbb{F}_q)}{P_{n,d}(\mathbb{F}_q)} - 1\right| \leq (n+1)(d-1)^n q^{-1}.$$

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Theorem I: Let $\delta := d_1 \cdots d_s$. There exist multihomogeneous pols

$$P_1, \ldots, P_{n-s+2} \in K[\Lambda^1, \ldots, \Lambda^s]$$

of degree $\deg_{\Lambda^i} P_j = \delta / d_i$ $(1 \leq i \leq s)$ such that

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- $P_1, \ldots, P_{n-s+2}$ form a regular sequence in $K[\Lambda^1, \ldots, \Lambda^s] \Rightarrow$ codimension at least $n - s + 2$.

Theorem II: Let $\delta := d_1 \cdots d_s$ and $\sigma := (d_1 - 1) + \cdots + (d_s - 1)$. There exist multihomogeneous polynomials

$$Q_1, \ldots, Q_{n-s+1} \in K[\Lambda^1, \ldots, \Lambda^s]$$

of degree $\deg_{\Lambda^i} Q_j = \delta\,(\sigma/d_i + 1)$ $(1 \le i \le s)$ such that

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Polynomials systems over K

Theorem III: Let $\delta := d_1 \cdots d_s$ and $\sigma := (d_1 - 1) + \cdots + (d_s - 1)$. There exist multihomogeneous polynomials

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- $R_1, \ldots, R_{n-s}$ form a regular sequence in $K[\Lambda^1, \ldots, \Lambda^s] \Rightarrow$ codimension at least $n - s$.

Polynomials systems over K

Theorem IV: Let $\delta := d_1 \cdots d_s$ and $\sigma := (d_1 - 1) + \cdots + (d_s - 1)$. There exist a multihomogeneous polynomial

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- Simpler approach which may be applied to families of polynomial systems.

Polynomials systems over $\mathbb{F}_q$

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Write $f_i := \sum_{\alpha \in \mathbb{N}_{d_i}^{n+1}} \lambda_\alpha^i X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ for $1 \leq i \leq s$, where $\lambda := (\lambda^1, \dots, \lambda^s) \in \mathbb{P}_{\mathbb{F}}^{D_1}(\mathbb{F}_q) \times \cdots \times \mathbb{P}_{\mathbb{F}}^{D_s}(\mathbb{F}_q) := \mathbb{P}_{\mathbb{F}}^{\mathbf{D}}(\mathbb{F}_q)$.

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Fact: for a multiprojective variety $W \subset \mathbb{P}_{\mathbb{F}}^{\mathbf{D}}$ of codimension k,

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- if $q > s(\sigma + n)\sigma^{n-s}\delta$, then there exists $\mathbf{f} \in N_{n,\mathbf{d}}(\mathbb{F}_q)$.

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Thank you for your attention!