A problem of Beelen, Garcia and Stichtenoth on an Artin-Schreier tower

Horacio Navarro

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Joint work with María Chara and Ricardo Toledano

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In this talk we show that the answer for this problem is negative for any s odd, i.e. the tower $\mathcal{H}$ is asymptotically bad over $\mathbb{F}_{2^s}$ for any s odd.

A motivation to study towers of function fields comes from coding theory.

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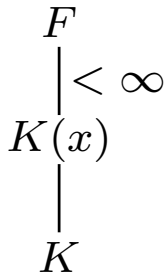

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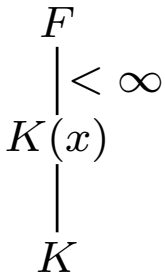

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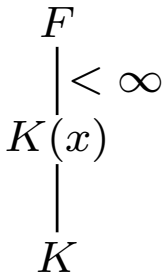

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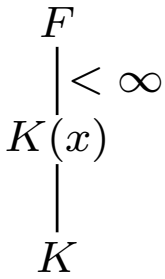

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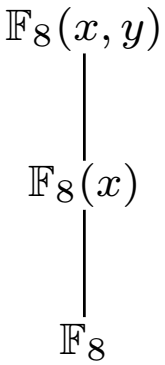

Towers of function fields

A *tower of function fields* over $\mathbb{F}_q$ is a sequence of algebraic function fields $\mathcal{F} = \{F_i\}_{i=0}^{\infty}$ such that

i) $F_i \subsetneq F_{i+1}$ for all $i \geq 0$,

ii) F_{i+1}/F_i is a separable finite extension for all $i \geq 0$,

iii) $\mathbb{F}_q$ is algebraically closed in F_i for all $i \geq 0$, and

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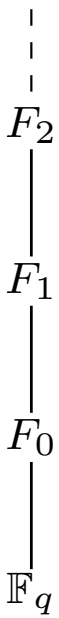

Figure : Sequence of function fields

A tower $\mathcal{F} = \{F_i\}_{i=0}^{\infty}$ over $\mathbb{F}_q$ is called *recursive* if there exist a sequence of transcendental elements $\{x_i\}_{i=0}^{\infty}$ over $\mathbb{F}_q$ and a bivariate polynomial $H(X, Y) \in \mathbb{F}_q[X, Y]$ such that $F_0 = \mathbb{F}_q(x_0)$ and

$$F_{i+1} = F_i(x_{i+1}), \quad \text{where } H(x_i, x_{i+1}) = 0,$$

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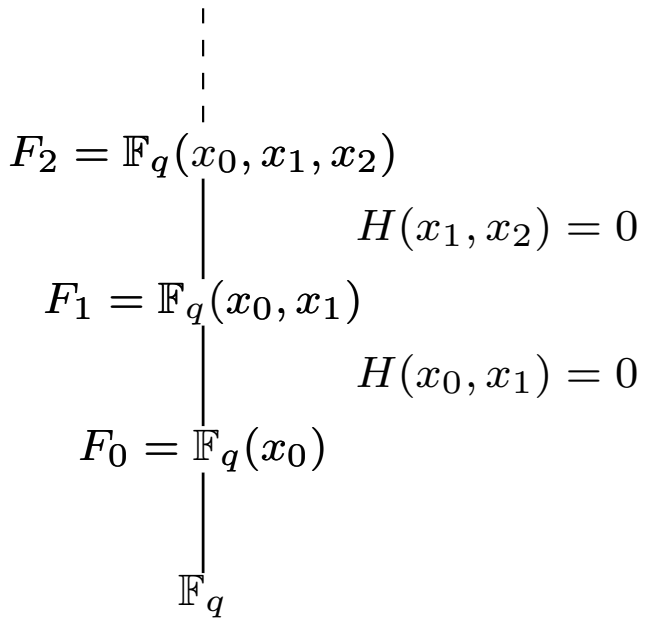

Figure : Recursive tower of function fields

The genus of $\mathcal{F} = \{F_i\}_{i \geq 0}$ over F_0 is defined as

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$\mathcal{F}$ is *asymptotically good* if $\lambda(\mathcal{F}) > 0$ (equiv. $\nu(\mathcal{F}) > 0$ and $\gamma(\mathcal{F}) < \infty$)

Theorem (Beleen-García-Stichtenoth).

Let $p = char(\mathbb{F}_q)$ and let $\mathcal{F}$ be a tower over $\mathbb{F}_q$ which can be described recursively by the equation $f(y) = g(x)$, with rational functions $f(t), g(t) \in \mathbb{F}_q(t)$ and $\deg(f(t)) = \deg(g(t)) = p$. Let $0 \neq e \in \mathbb{F}_q$ and

$$\mathrm{P}(t) := t^p - e^{p-1}t \in \mathbb{F}_q[t].$$

Suppose that $\mathbb{F}_q(x,y)/\mathbb{F}_q(x)$ and $\mathbb{F}_q(x,y)/\mathbb{F}_q(y)$ are Galois extensions. Then $\mathcal{F}$ can be described by one of the following equations:

$$\mathrm{P}(y) = \begin{cases} \frac{a}{\mathrm{P}(\alpha x)+b} + c & \text{(case 1)} \\ a\,\mathrm{P}(\alpha x) + b & \text{(case 2)} \\ \frac{a}{\mathrm{P}(\alpha/x)+b} + c & \text{(case 3)} \end{cases}$$

with $a, \alpha \in \mathbb{F}_q^*$, $b, c \in \mathbb{F}_q$. In case 1 $\alpha \notin \mathbb{F}_p$ and $a \notin \mathbb{F}_p$ in case 2.

Remark.

Let $\mathcal{F}$ be a tower over $\mathbb{F}_2$ which can be described recursively by the equation

$$f(y) = g(x),$$

where $f(t), g(t) \in \mathbb{F}_2(t)$ are rational functions such that $\deg(f(t)) = \deg(g(t)) = 2$. Then $\mathcal{F}$ can be described by

$$y^2 + y = \frac{1}{(1/x)^2 + (1/x) + b} + c$$

with $b, c \in \mathbb{F}_2$.

b	c	equation	asymptotically
0	0	$y^2+y=\dfrac{x^2}{x+1}$	good over $\mathbb{F}_4$
0	1	$y^2+y=\dfrac{x^2+x+1}{x+1}$	good over $\mathbb{F}_8$
1	0	$y^2+y=\dfrac{x^2}{x^2+x+1}$	good over $\mathbb{F}_8$
1	1	$y^2+y=\dfrac{x+1}{x^2+x+1}$	?

The genus of the tower

Theorem.

Let s be a positive integer. The tower $\mathcal{H} = \{F_i\}_{i=0}^{\infty}$ recursively defined over $\mathbb{F}_{2^s}$ by the equation

$$y^2 + y = \frac{x}{x^2 + x + 1} \tag{1}$$

has finite genus, more precisely

$$\gamma(\mathcal{H}) = \lim_{i \to \infty} \frac{g(F_i)}{2^i} \leq 4$$

The splitting rate of the tower

Theorem.

Consider the tower $\mathcal{H} = \{F_i\}_{i=0}^{\infty}$ over $\mathbb{F}_{2^s}$ with s odd. Then the number of rational places of F_i is $2(|S|+1)$ for all $i \geq 1$ where

$$S = \left\{ \alpha \in \mathbb{F}_{2^s} : \mathrm{Tr}\left(\frac{\alpha}{\alpha^2 + \alpha + 1} \right) = 0 \right\}.$$

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Theorem.

The tower $\mathcal{H} = \{F_i\}_{i=0}^{\infty}$ over $\mathbb{F}_{2^s}$, with s odd, has splitting rate zero, i.e.

$$\nu(\mathcal{H}) = \lim_{i \to \infty} \frac{N(F_i)}{2^i} = 0.$$

Lemma (The key lemma)**.**

Let $\theta, \beta \in \mathbb{F}_{2^s}$ with s odd, such that $\theta^2 + \theta = \frac{\beta}{\beta^2+\beta+1}$. Then

$$\mathrm{Tr}\left(\frac{\theta}{\theta^2+\theta+1}\right) \neq \mathrm{Tr}\left(\frac{\theta+1}{\theta^2+\theta+1}\right).$$

Proof.

Suppose that

$$\mathrm{Tr}\left(\frac{\theta}{\theta^2+\theta+1}\right) = \mathrm{Tr}\left(\frac{\theta+1}{\theta^2+\theta+1}\right) \text{ then } \mathrm{Tr}\left(\frac{1}{\theta^2+\theta+1}\right) = 0.$$

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On the other hand, by hypothesis $\theta^2+\theta = \frac{\beta}{\beta^2+\beta+1}$ then

$$\frac{1}{\theta^2+\theta+1} = \frac{\beta^2+\beta+1}{\beta^2+1} = 1 + \frac{\beta}{\beta+1} + \left(\frac{\beta}{\beta+1}\right)^2.$$

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Finally, since $\mathrm{Tr}(1) = 1$ and $\mathrm{Tr}(\alpha) = \mathrm{Tr}(\alpha^2)$ for all $\alpha \in \mathbb{F}_{2^s}$, we have

$$\mathrm{Tr}\left(\frac{1}{\theta^2+\theta+1}\right) = \mathrm{Tr}(1) + \mathrm{Tr}\left(\frac{\beta}{\beta+1}\right) + \mathrm{Tr}\left(\left(\frac{\beta}{\beta+1}\right)^2\right) = 1,$$

a contradiction.

$$
\begin{array}{l}
\vdots \\
F_3 = \mathbb{F}_{2^s}(x_0, x_1, x_2, x_3) \\
\Big| \\
F_2 = \mathbb{F}_{2^s}(x_0, x_1, x_2) \qquad \phi_2(t) = t^2 + t + \frac{x_1}{x_1^2 + x_1 + 1} \\
\Big| \\
F_1 = \mathbb{F}_{2^s}(x_0, x_1) \qquad \phi_1(t) = t^2 + t + \frac{x_0}{x_0^2 + x_0 + 1} \\
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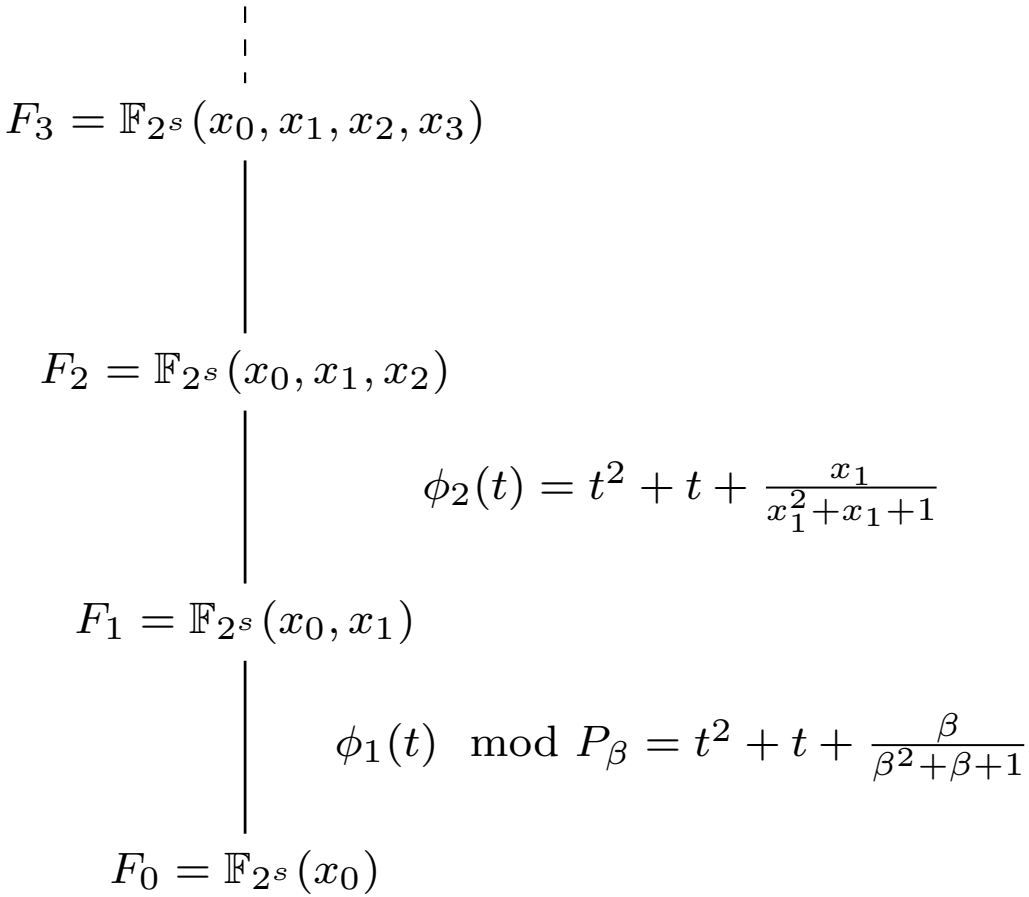
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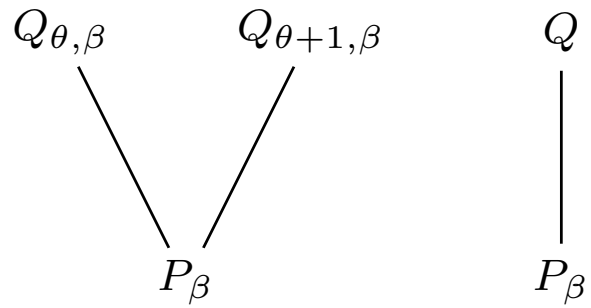
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P_β
Q
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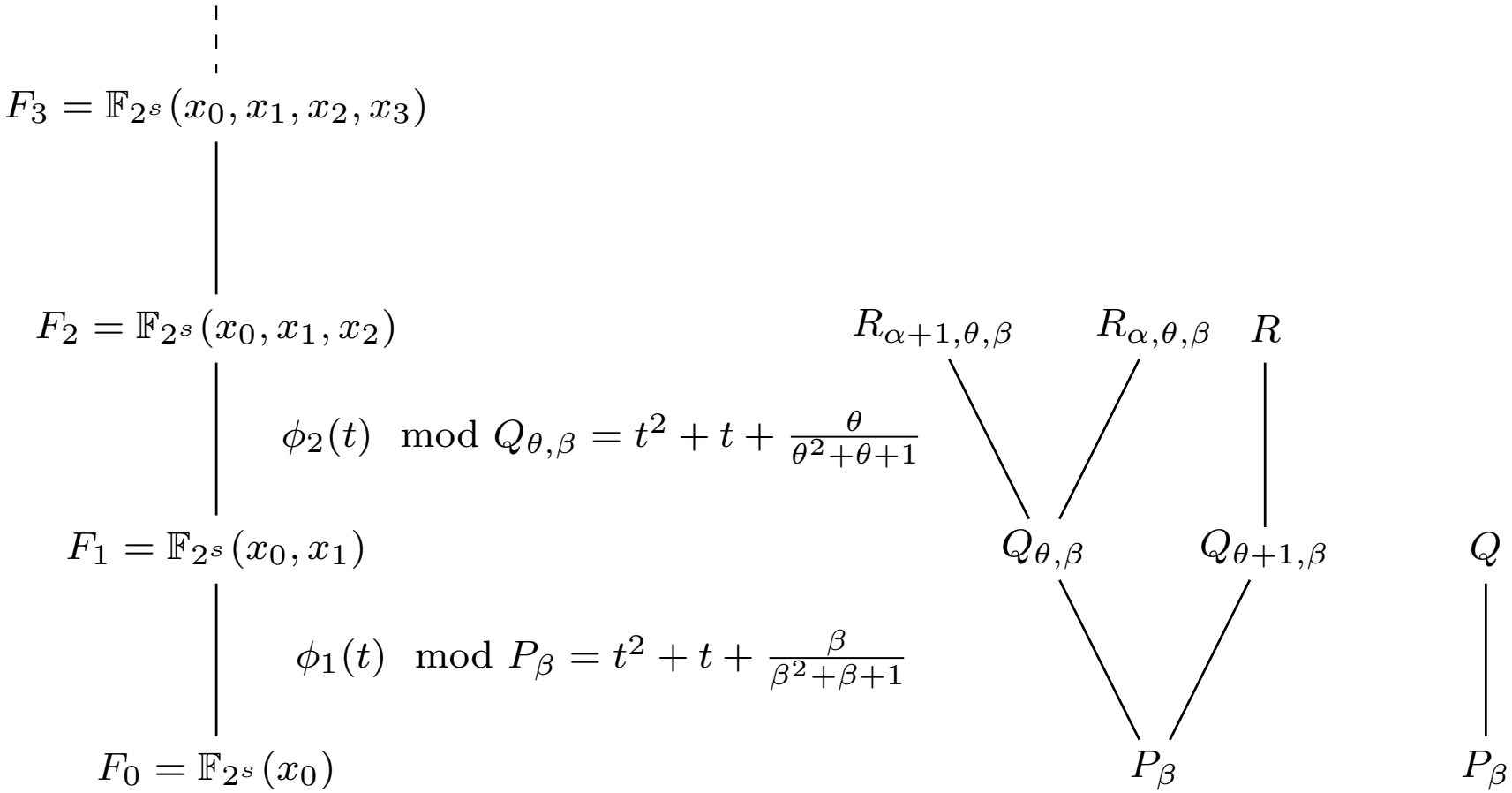
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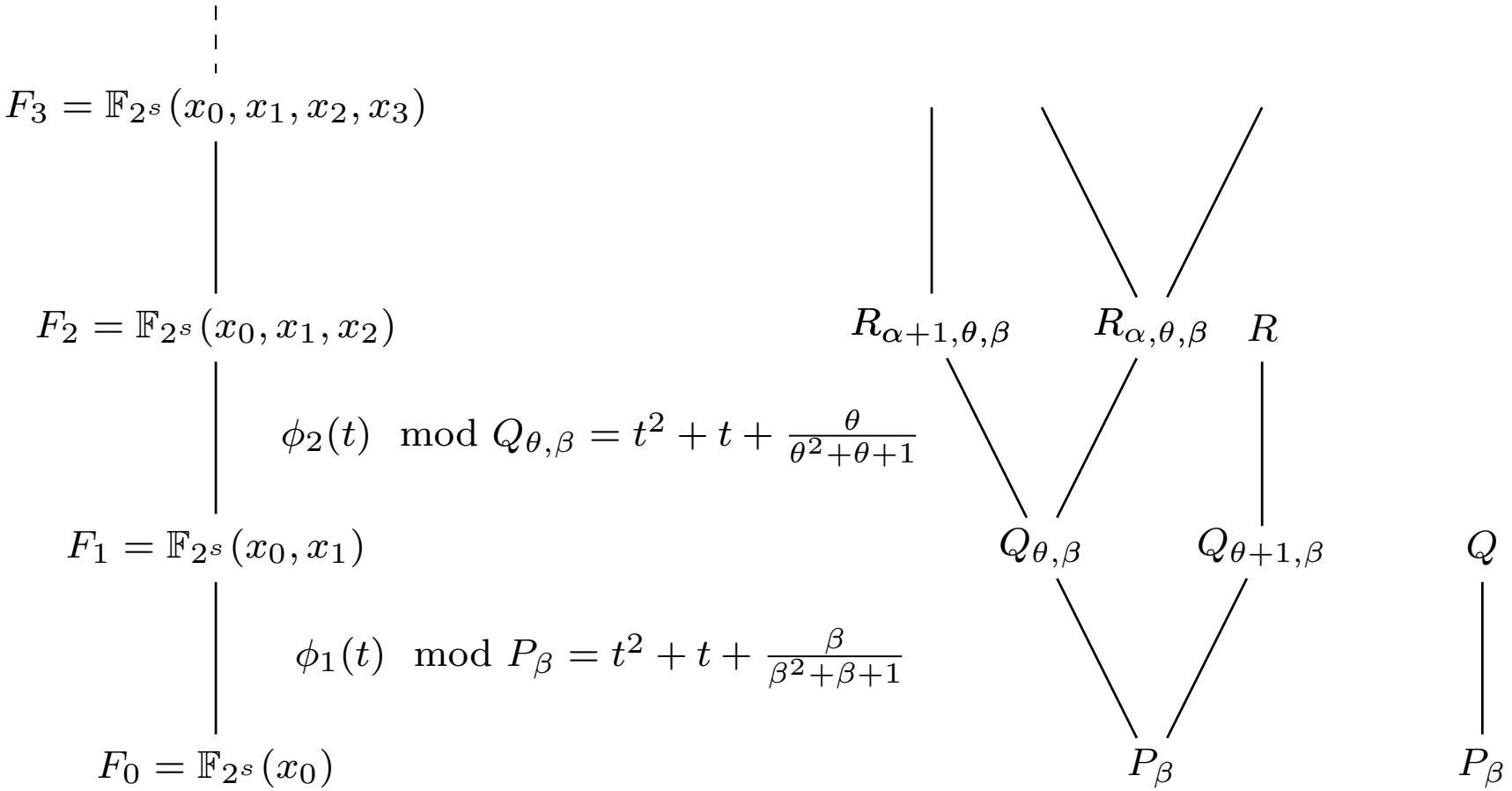
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