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NON-CONCAVE MULTIFRACTAL SPECTRA WITH WAVELET LEADERS PROJECTION OF SIGNALS WITH AND WITHOUT CHIRPS

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Abstract

The multifractal spectrum has been revealed recently as a very useful tool for the analysis of signals, especially in those coming from the measurement of physical variables in chaotic or critical systems with multifractal structure, and in which many phenomena interact in multiple scales. In addition, in this context, the presence of singularities of rapid oscillation, chirps, is frequently difficult to handle. In this work we propose a way for using "wavelet leaders", avoiding the Legendre transform, for developing a new method capable of recognizing if a non-concave multifractal spectrum arises in a given signal, and is also capable of obtaining it in the case in which oscillating singularities appear, and so it succeeds where any other method fails.

Keywords: Multifractal Formalisms; Non-Concave Spectra; Chirps; Wavelet Leaders; GMWLP.

1. INTRODUCTION

Multifractal analysis emerged in the mid 80's with the works of Mandelbrot,[1] Parisi, Frisch, Arneodo and others in the context of modelling fully developed turbulence.[2] Since then it has arisen as a relevant tool for the study and characterization of

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many natural series. Multifractals have been used to describe phenomena from physics,[3] economics,[4] physiology,[5–7] meteorology[8] and many other fields.[9,10]

Basically it consists of the analysis of the dimension of the sets of each degree of pointwise Hölder regularity, notwithstanding some other notions of regularity were investigated (local Hölder regularity, instead of pointwise, p-exponent, and others). Additionally which type of fractal dimension must be taken is a delicate matter related with the type, strong or weak, of the singularities under consideration.

Another question to be considered is the recognition of the Hölder exponents when the signals have oscillating singularities, like chirps, in addition to the cusp singularities that are easier to handle.

Most of the usual methods[11] for computing multifractal spectra are unstable in the face of the presence of chirps, sometimes because the impossibility of obtaining reliable estimates of the trends when rapid oscillation occurs and some other times because the dispersion of the maxima lines of wavelet coefficients. Jaffard's method of Wavelet Leaders ("WL") was until today, perhaps the only one that is not affected seriously by the presence of chirps.

Frequently another misunderstood matter is the possibility of the non-concavity of the multifractal spectra, and that is perhaps a nice problem to be studied. The pointwise estimation of Hölder exponents requires some kind of limit process. The majority of the methods avoids the problem about the numerical instabilities of this process by means of some average quantities related to a scaling function from certain p-norms and by obtaining the spectrum provided by the Legendre transform of that scaling function. But this procedure imposes the concavity feature for the spectra so obtained, reinforcing the prejudice mentioned in some works that the concavity might be inherent for natural series. On the other hand, the results of some assays that we did with series from EEG of epileptic crisis of brain absences suggest the idea that non-concave spectra arise naturally in some contexts.[12] Actually, if we take in consideration that the series of each channel of EEG proceed from the electrical activity of great groups of neurons and, because the temporal and spatial evolution of the epileptic crisis, many of these neurons may have a kind of multifractal behavior, and many others may have another different behavior notably displaced from the first one. It is not surprising that non-concave spectra appears. And, of course, it is quite credible that many physical systems would exhibit non-concave spectra if we were able to detect them.

How can we detect them if the methods themselves impose the concavity? A few algorithms based in density estimation of Hölder exponents, the most successful of them is perhaps the Gradient Modulus Wavelet Projection ("GMWP", Turiel *et al.*[13]), may show non-concave multifractal spectra. The latter is probably preferable over many others because of the greater stability of the estimation provided by the wavelet coefficients. But these methods suffer the above mentioned problems if there are oscillating singularities.

How to take advantage of the success of each method?

The natural response seems to be replacing the wavelet coefficients by the wavelet leaders and not obtaining the spectra by means the Legendre Transform. Instead, we compute the dimension of Hölder exponents sets from the densities of them with the GMWP. So we are able to preserve the possibility of having non-concave spectra, in disagreement with the methods that use Legendre Transform and cannot calculate them. On the other hand, in this way, the stability in presence of chirps of the signals is preserved by the leaders, in opposition to GMWP. We called this hybrid method: "GMWLP" (Gradient Modulus Wavelet Leaders Projection), and in this work, we compare its performance against the WL and the GMWP in many natural and synthetic series with and without chirps and with concave and non-concave spectra.

2. THEORETICAL FOUNDATIONS

2.1. Multifractality and Hölder Exponents

The starting point to analyze irregular signals is the *Hölder regularity*, used to calculate the *singularity spectrum*. This *spectrum* describes fractal properties of the given signal. Let us give a brief overview of these concepts.[14–16]

Let $g(x)$ be a continuous function. For any non-integer $\alpha > 0$, $g(x) \in C^{\alpha}_{x_0}$, *the pointwise Lipschitz class*, if there exists a constant $c > 0$ such that close to the point x_0

$$|g(x) - P(x - x_0)| \leq c|x - x_0|^{\alpha},$$

where P is a polynomial of degree less than or equal to the integer part of α. The pointwise exponent

Hölder of g at x_0, $h_g(x_0)$ is the supremum of α, such that $g \in C^{\alpha}_{x_0}$.

Cusp singularities are related to non-integer exponents and the singular points may be isolated or may accumulate in dense sets. Moreover, in the last case the Hölder exponents can vary from point to point, providing very complex structures for the respective signal. Then, the natural description of these structures is the distribution of the exponents in the regularity range. With more precision, we can consider measures of the sets of points having the same Hölder exponent. Since in general these sets have zero Lebesgue measure, although being uncountable sets, fractal measures are the appropriate tool to deal with them.

The *Hölder singularity spectrum* $f(\alpha)$ is defined[17] as the Hausdorff dimension of the set of points x_0 having the exponent $h_g(x_0) = \alpha$. We must remark that in practice, this dimension is replaced by others that are analogous, such as the *box-dimension*, more malleable for the numerical applications.[14]

Then, the main problem is to estimate the singularity *spectrum* from the data values. Since the numerical application of the mathematical definition is almost impossible, some additional hypothesis or a more specific framework are needed.[18]

In this way, most empirical methods are based on the so-called *multifractal formalism*.[19] This essentially consists of a proposed *structure function*, $F_q(l)$ providing global estimations about the regularity of the signal as a function of a semi-norm parameter q and the scale paramater l, supposedly related with the pointwise behavior in this way

$$f(\alpha) = \inf_q (q\alpha - h(q) + 1)$$

for a concave function h, assuming that

$$F_q(l) \sim |l|^{h(q)}.$$

An illustrative example for the structure function is given by Ref. 4

$$F_q(l) = \int_R |g(x+l) - g(x)|^q \, dx.$$

Multifractal formalisms are open for a broad family of structure functions and related numerical implementations. Let's observe that the above given formula can be discretized or replaced by other difference operators. In particular we can replace the scale parameter l as follows

$$F_q(s) = \int_R |g(s(x+1)) - g(sx)|^q \, dx,$$

then, for $\tau = h - 1$,

$$f(\alpha) = \inf_q (q\alpha - \tau(q))$$

and the scale operator becomes evident.

We will discuss and revise several alternatives below. But let us remark that, except for some special cases, there is not a theoretical proof for these empirical methods. In particular local self-similarity structures seem to be the necessary ingredient for multifractal formalisms. Beyond these hypotheses we cannot ensure the validity of the methods based on the formalism, and multifractal estimation can be considered as an open problem.[16]

2.2. The Wavelet Leaders Method

Until recently the principal Wavelet based method for estimating the Multifractal Spectrum was the Wavelet Transform Modulo Maxima (WTMM).[20] This was usually preferred over the Wavelet Integral Transform Method and also over the methods based in the q-moduli of continuity, a predecessor of the MFDFA, because it exhibits better numerical properties (see Bacry *et al.*,[21] Jaffard[15]) and because, at least in the cases of self-similarity and statistical self-similarity, we can theoretically ensure that the wavelet-based Multifractal Formalism gives the correct spectrum. Moreover the WTMM can be used for signals with a significant fraction of null values of the series where MF-DFA cannot be used (see Kantelhardt *et al.*[22]). This case is very frequent in monofractal series. Nevertheless the mentioned method is not free of criticisms: For series of fractional Brownian motion the part of the spectrum corresponding to the highest regularity, that obtained from the negative values of q, is underestimated (see, e.g. Jaffard *et al.*[23] or Oswiecimka *et al.*[24]). It seems that the source of this trouble comes from the fact that WTMM actually captures the weak exponent spectrum, which sometimes coincides with the Hölder exponent. The step to higher dimensions increases enormously the complexity when working the WTMM with the continuous wavelet transform.

Additionally the application of both methods, WTMM and MF-DFA, presents difficulties when the signal has oscillating singularities.

An interesting proposal that overcomes the majority of the problems of both methods is the use of the Wavelet Leaders:

Wavelets[25] are natural tools in multifractal analysis for at least three reasons. First, the concept of self-similarity is implicit in the construction of the wavelet basis $\{\Psi_{j,k}\}$ that we define below. Second, wavelet coefficients provide a time-scale decomposition of the initial function (or signal) f, hence, scaling properties of a function shall imply scaling properties of its wavelet coefficients. Finally, the pointwise Hölder exponent $h_f(x)$ of any continuous function f around a point x can be computed through size estimates of the wavelet coefficients $c_{j,k}$ associated with f. Thus, they are efficient tools to analyze local behaviors.[26,27]

Jaffard[28] developed a method for the wavelet characterization of the pointwise Hölder exponent and the relationship between Hölder regularity and local oscillation. He gave a new formulation of this criterion in terms of local suprema of the wavelet coefficients that he named Wavelet Leaders, which paves the way for the new multifractal formalism. It can be summarized as follows:

For any pair $(j,k) \in N \times Z, I_{j,k}$ denotes the dyadic interval $[k2^{-j},(k+1)2^{-j})$.

Then, if $x \in R$, $\forall j \geq 1$, there exists a unique integer $k_{j,x}$ such that $x \in I_{j,k_{j,x}}$.The interval $I_{j,k_{j,x}}$ is also denoted $I_j(x)$.

Let a wavelet $\Psi \in C^{\infty}(R)$ be a function in the Schwartz class, as constructed in Meyer (1990),[29] then the set of functions $\{\Psi_{j,k} = \Psi(2^j \cdot - k)\}_{(j,k)\in Z^2}$ forms an **orthogonal** wavelet basis (owb) of $L^2(R)$. Also, let $f \in C^{\epsilon}(R)$ for some $\varepsilon > 0$, and we write f in the owb

$$f = \sum_{j,k\in Z} c_{j,k}\Psi_{j,k}, \tag{1}$$

where $c_{j,k}$ is the wavelet coefficient of f defined by

$$c_{j,k} := c_{j,k}(f) = 2^j \int_R f(t)\Psi_{j,k}(t)dt. \tag{2}$$

For any pair $(j,k) \in N \times Z$, let us introduce the Wavelet Leader $L_{j,k}$ associated with f and Ψ

$$L_{j,k} := \sup_{j' \geq j, k'2^{-j'} \in I_{j,k}} |c_{j',k'}| \ . \tag{3}$$

Then, with any point $x_0 \in R$ and any scale $j \geq 0$ can be associated the coefficient

$$L_j(x_0) := \sup_{|k-k_{j,x}|\leq 1} L_{j,k} \ . \tag{4}$$

A theorem about the pointwise Hölder exponent proved in Jaffard[28] affirms that

$$h_f(x_0) = \liminf_{j\to+\infty} \frac{\log(L_j(x_0))}{\log(2^{-j})} \tag{5}$$

Recall that the Legendre transform of a function $\varphi : q \in R \mapsto \varphi(q)$ is the mapping

$$\begin{aligned}\varphi^* : h \in R &\mapsto \varphi^*(h) \\ &= \inf_{q\in R}(qh - \varphi(q)) \in R \cup \{-\infty\}.\end{aligned} \tag{6}$$

For any function $f \in L^2(R)$ decomposed in Eq. (1), one can introduce the scaling function ξ_f associated with f

$$\begin{aligned}\xi_f : p \in R &\mapsto \xi_f(p) \\ &= \liminf_{j\to+\infty} - j^{-1}\log_2\left(\sum_{k\in Z}{}^{*}|L_{j,k}|^p\right),\end{aligned} \tag{7}$$

where $*$ means that the sum is taken over the k such that $|L_{j,k}|$ does not vanish. For each $j \geq 1$ the function $p \mapsto \sum_{k\in Z}{}^{*}|L_{j,k}|^p$ is log-convex and non-increasing when j is large enough. In that case, the mapping ξ_f is concave and non-decreasing on R (as limit of the infimum of non-decreasing concave functions).

This kind of free energy function is naturally introduced in order to formulate a multifractal formalism for functions based on the representation as wavelet series (see Jaffard (2007)).[28] Frisch and Parisi first proposed[30] a formula that links the multifractal spectrum of a function f with some averaged quantities derived from f. This formula, generally called Frisch–Parisi's conjecture, can be generalized and reformulated.[30] If we call $d_f(h)$ to the Hausdorff dimension of the set $E_h^f := \{x : h_f(x) = h\}$, then

$$d_f(h) = \inf_{p>0}(ph - \eta_f(p)) = (\eta_f)^*(h), \tag{8}$$

where the mapping $\eta_f : p \in R \mapsto R \cup \{-\infty\}$ is a suitable free energy function associated with f.

Then, Jaffard[28] establishes that the scaling function ξ_f depends only on f, NOT on Ψ, and if f satisfies the multifractal formalism at the exponent $h > 0$, $\eta_f = \xi_f$. Finally, we build the Hausdorff Multifractal Spectrum of f defined by

$$d_f : h \mapsto d_f(h) = \dim(E_h^f) = (\xi_f)^*(h). \tag{9}$$

2.3. Hölder Estimation by Means Wavelet Leaders

The pointwise Hölder exponents of a function can be estimated studying the decay of the wavelet leaders in the cone of influence of every point, i.e. the set of dyadic intervals $I_j(x)$ and their adjacent intervals. Actually the characterization of spaces of functions by means of the use of the wavelet transform is well known. It can be obtained by the following bounds[28,31] for the wavelet coefficients of f, if $f \in C^\alpha(x_0)$

$$|c_{jk}| \leq C 2^{-\alpha j}(1 + |2^j x_0 - k|)^\alpha \tag{10}$$

for some constant C, and with a weaker reciprocal (there appears a factor of logarithmic decay involved). In this line, under general enough hypotheses it is possible to obtain the pointwise Hölder exponent if one assumes that

$$|c_{jk}| \approx K 2^{-H_f(x_0)j}$$

since we obtain: $\log_2 |c_{jk}| \approx \log_2 K - H_f(x_0)j$, and then

$$\lim_{j\to+\infty} \frac{\log_2 |c_{jk}|}{-j} = H_f(x_0).$$

The above mentioned procedure — besides of the specifies presumptions that its application needs on a discreet series — suffers from disadvantages when the signal shows oscillating singularities (e.g. chirps). When a time-frequency singularity lies on small intervals, great values of K are necessary, and then they cannot be neglected even at the maximum scale that the discretization admits. This cannot be avoided since the constants involve (after fixing a scale) both C and the factor $(1 + |2^j x_0 - k|)^\alpha$, and this introduces instabilities in these estimations.

On the other hand, by considering only the wavelet leaders, the bounds are unified in all the intervals of the cone of influence — that alludes also to the contiguous intervals in every scale close to x_0-, and then perturbations introduced by the oscillations decrease. This allows to formulate the inequality of the law of decay without the factor that involves the translations

$$|L_{jk}| \leq C 2^{-\alpha j}$$

(and also there is a reciprocal with a logarithmic factor analogous to the case of the wavelet coefficients). Then we do an extrapolation to the limit to estimate $H_f(x_0) = \lim_{j\to+\infty} \frac{\log_2 |L_{jk}|}{-j}$. This provides a faithful estimation of the pointwise Hölder exponents that does not produce great variations in the presence of chirps of the signal.

2.4. Gradient Modulus Wavelet Leaders Projection Method

This method combines the use of the Wavelet Leaders with an interesting idea of Turiel *et al.*[13] that essentially consists of estimating the multifractal spectrum directly from a histogram of the Hölder exponents deduced by extrapolating the behavior of the wavelet coefficients, the so called Gradient Modulus Wavelet Projection (GMWP).

Turiel *et al.*[13] use an histogram to estimate the density $\rho(\alpha)$ of the Hölder exponents α from the values for the smallest scale j considering

$$c_{j,k} := c_{j,k}(f) = 2^j \int_R f(t)\Psi_{j,k}(t)dt,$$
$$\text{for } \varepsilon_0 = 2^{-j_0} \ll 1.$$

The $\Psi_{j,k}$ are not necessarily a base of L^2 and they may have no vanishing moments. The idea of Turiel is to consider the variation of the projections of the signal against a family of functions that change its scale (because his objective is to estimate the α of the cusp type singularities).

The first observation that arises is the possibility of improving the estimations by extrapolating to the limit, instead of using small values of the scale.

Also, the projections against a family of functions parametrized according to the scale are not always comparable with $|x - x_0|^\alpha$, because for this it is necessary to consider that if $\Psi_{j,k}$ has the first null moment, then $\int_{\mathbb{R}} f(t_0)\Psi_{j,k}(t)dt = 0$, hence

$$\begin{aligned}\left|\int_{\mathbb{R}} f(t)\Psi_{j,k}(t)dt\right| &= \left|\int_{\mathbb{R}} (f(t) - f(t_0))\Psi_{j,k}(t)dt\right| \\ &\leq \int_{\mathbb{R}} |f(t) - f(t_0)||\Psi_{j,k}(t)|dt \\ &\simeq \int_{\mathbb{R}} |t - t_0|^{H_f(t_0)}|\Psi_{j,k}(t)|dt.\end{aligned}$$

Using this with a hypothesis on the decay of $\int_{\mathbb{R}} |\Psi_{j,k}(t)|dt$, we obtain the condition $|L_{jk}| \leq C2^{-\alpha j}$, if $0 < H_f(t_0) < 1$.

If $H_f(t_0) > 1$, we seek conditions that imply

$$|f(t) - P(t - t_0)| \simeq |t - t_0|^{H_f(t_0)}$$

(where $P(t - t_0)$ is a Taylor polynomial) and then it is necessary that

$$\int_{\mathbb{R}} t^k \Psi_{j,k}(t)dt = 0, \quad \text{for } k \leq \deg(P).$$

So, we cannot ignore the requirement of several vanishing moments for the wavelet.

Finally, it is necessary to define with better precision the density $\rho(\alpha)$ function of α-singularities.

On the other hand, working with the behavior of the wavelet coefficients c_{jk} implicitly assumes that there are only cusp singularities, discarding the presence of chirps (see the next section).

Our proposal is to estimate with a direct method, similar to that mentioned above, based on the behavior of wavelet leaders L_{jk} instead of the c_{jk}.

In this way the stability in the face of the oscillating singularities provided by the wavelet leaders is preserved. Moreover we also have the freedom of the method GMWP concerning the concavity or non-concavity of the multifractal spectrum.

Then we will use the abbreviation GMWLP (Gradient Modulus Wavelet Leaders Projection), for the algorithm of direct estimation of the spectrum with wavelet leaders, which consists of these steps:

- We estimate the pointwise Hölder exponents by the limit
$$j = N, \frac{N}{2}, \frac{N}{2^2}, \ldots, \frac{N}{2^{\log_2 N}} \to 0 : H_f(x_0) = \lim_{j\to+\infty} \frac{\log_2 |L_{jk}|}{-j}.$$
- We estimate the density $\rho(\alpha)$.
- Theoretically $\rho(\alpha)$ is the density, that is the derivative of the distributon function F such that $F(\alpha) = P(x \le \alpha) = \int_{-\infty}^{\alpha} \rho(t)dt$. And then:
$$\rho(\alpha) = \frac{d(F(\alpha))}{d\alpha} = \frac{d(P(x \le \alpha))}{d\alpha} = \lim_{\varepsilon\to 0} \frac{P(x \le \alpha+\varepsilon) - P(x \le \alpha-\varepsilon)}{2\varepsilon}.$$
So we take a partition of $[\alpha_{\min}, \alpha_{\max}]$ as finely as possible and we count the proportion of the points of the series with Holder exponent lying in each interval:
$$\rho(\alpha) \approx \frac{1}{2\varepsilon}\left(\frac{\#\text{ of points with } \alpha-\varepsilon \le H_f(x) \le \alpha+\varepsilon}{\#\text{ total of the series}}\right)$$
- We estimate dimension for each α: $\dim(E_\alpha) = \lim_{r\to 0} \dim(E_{\alpha_1}) - \frac{\log(\frac{\rho(\alpha)}{\rho(\alpha_1)})}{\log(r)}$ where E_{α_1} is set of maximum density and r is the ratio of the intervals of the mesh:

Frequently the series has full support (dimension 1) and then the set E_{α_1} of maximum density must also has dimension 1.

On the other hand for the smallest scales ($r \approx 0$) the approximation:
$$\dim(E_\alpha) \approx 1 - \frac{\log(\frac{\rho(\alpha)}{\rho(\alpha_1)})}{\log(r)} = \dim(E_{\alpha_1}) - \log_r\left(\frac{\rho(\alpha)}{\rho(\alpha_1)}\right)$$
is equivalent to $r^{\dim(E_\alpha)} \approx \frac{r^{\dim(E_{\alpha_1})}}{\frac{\rho(\alpha)}{\rho(\alpha_1)}}$, and consequently $\frac{\rho(\alpha_1)}{\rho(\alpha)} \approx \frac{r^{\dim(E_\alpha)}}{r^{\dim(E_{\alpha_1})}}$, as one expect for any reasonable definition of dimension.

3. WAVELET LEADERS VERSUS OTHER METHODS IN SIGNALS WITH CHIRPS

Besides the advantages of the WL in terms of algorithmic complexity — only $N \cdot \log N$ products are necessary — and efficiency for monofractal signals (this is a lack for many statistical methods: DFA and MF-DFA for instance), a decisive superiority of the WL is the fidelity in signals presenting chirps.

In the next section we will exhibit examples of comparison of the mentioned methods. With series that do not present chirps we will see that the results are similar. On the other hand, in series obtained from the first ones by adding chirps with C^∞ regularity, in almost every point the real multifractal spectrum doesn't vary; for the WL method the spectrum does not suffer very sensitive distortions, notwithstanding the results obtained using MF-DFA show substantial alterations.

We will discuss now an heuristic justification for the superior efficiency of the methods based in wavelet leaders in signals with oscillating singularities.

The following result is due to Jaffard.[28] Let $\alpha > 0$. If $f \in C^\alpha(x_0)$, then there exists $C > 0$ such that $\forall j \ge 0$, $L_j(x_0) \le C2^{-\alpha j}$. Conversely if $\forall j \ge 0$, $L_j(x_0) \le C2^{-\alpha j}$ then $f \in C^\alpha_{\log}(x_0)$.

The requirement of belonging to $C^\alpha_{\log}(x_0)$ is just a little weaker than belonging to $C^\alpha(x_0)$. Actually, $f \in C^\alpha_{\log}(x_0)$ if and only if $\exists C, \delta > 0$ and a polynomial P of degree at most $[\alpha]$, the integer part of α, such that: if $|x - x_0| \le \delta$, then $|f(x) - P(x-x_0)| \le C|x-x_0|^\alpha \log(\frac{1}{|x-x_0|})$. Clearly $C^\alpha(x_0) \subset C^\alpha_{\log}(x_0) \subset C^{\alpha-\varepsilon}(x_0)\ \forall\varepsilon > 0$.

For that reason, working with discreet data we can assume that the condition of belonging to

$C^\alpha(x_0)$ is essentially equivalent to $L_j(x_0) \asymp C2^{-\alpha j}$, that is, $L_j(x_0)$ is asymptotic to $C2^{-\alpha j}$.

Then, let's suppose that we have : $L_j(x_0) = C2^{-\alpha j}$, from some scale j_0 on and with respect to an orthogonal wavelet ψ with enough null moments and belonging to the Schwartz's class, (except of an error of higher order). For simplicity we consider the case $0 < \alpha < 1$; then $P \equiv f(x_0)$. Let's see what happens when adding a chirp: A chirp essentially corresponds to a diagonal region for the time-frequency plane, corresponding to an almost asymptotic curve — say in x_0 — for the time-scale plane with respect to the $\|\cdot\|_\infty$ norm (detailed rigorous approaches to chirps are given by Jaffard and Meyer[31]). A simple example of a chirp is $f(x) = |x|^\alpha \sin(\frac{1}{|x|^\beta})$ for $x_0 = 0$.

Let $Ch(\cdot)$ be an ideal chirp at x_0, that is a chirp such that the corresponding wavelet coefficient with respect to ψ is $c_j(Ch)(x_0) = (-1)^j 2^{-j}$. Because of the vanishing moments the coefficients for each scale will be 0 and then:

$$L_j(f)(x_0) \le C2^{-\alpha j} + |c_j(x_0)|$$
$$= C2^{-\alpha j} + 2^{-j} = O(2^{-\alpha j}).$$

Nevertheless, actually the situation will never be ideal. The probability of having a point x_0 such that the supports of the dilations of the wavelet for each scale were centered perfectly around x_0 is virtually zero. Consequently we will have troubles: Since the absolute values of the wavelet coefficients in x_0, $|c_j(x_0)|$, may be considerably less than $C2^{-\alpha j}$ for certain levels, and because of the terms $(-1)^j 2^{-j}$ added due to the chirp, the values of $\log_2(|c_j(x_0)|)$ will move across the regression line, seriously disturbing the estimation of the slope.

On the other hand, when considering the intervals contiguous to the cone of influence of x_0 for each scale, we find that the cancellations due to the moments won't occur in some of them. So, we will often have for each j that some of the coefficients considered for the calculation of the leader will be of order $2^{-\alpha j}$, and also $L_j(x_0)$ will be $O(2^{-\alpha j})$. And then, we will have for the leaders: $O(2^{-\alpha j}) + O(2^{-j}) = O(2^{-\alpha j})$. Evidently there can be slight deviations for some scale and for small values of j, but not asymptotically (this is the moral of the mentioned theorem of Jaffard). This way a prudent implementation of the WL will provide right values for the terms of the partition function.

One must notice that when considering only the wavelet coefficients there will be cancellations almost everywhere for arbitrarily great scales. So, by incorporating chirps, there will always be unpredictable jumps across the line of regression for the points close to the location of maximal frequency. And then the corresponding terms of the partition function will be substantially affected. Thus, when incorporating a few chirps whose regions of high frequency cover, for the corresponding level of discretization, a substantial part of the support of the signal, there would be no certainty at all about of the accuracy of the formalisms based in wavelet coefficients instead of wavelet Leaders. On the other hand, the addition of an infinitely oscillating function, if it has zero-mean, does not alter the profiles, but the trends cannot be eliminated with polynomial fitting despite the order of the MF-DFA used. So the estimation with this method also turns out to be completely ruined though there are few chirps in the signal.

4. TESTS FOR SERIES WITH AND WITHOUT CHIRPS WITH CONCAVE AND NON-CONCAVE SPECTRA

The performance of both methods is compared (Figs. 1 and 2) for random wavelet cascades and for binomial multifractal series (BMS), with known multifractal behavior. For BMS the parameter a, that we will describe later, goes from 0.52 up to 0.98 (with step 0.01), the admissible values of a lie in the interval $(0.5, 1)$. We avoid values close to 0.5 or 1 in order to elude numerical instabilities. For

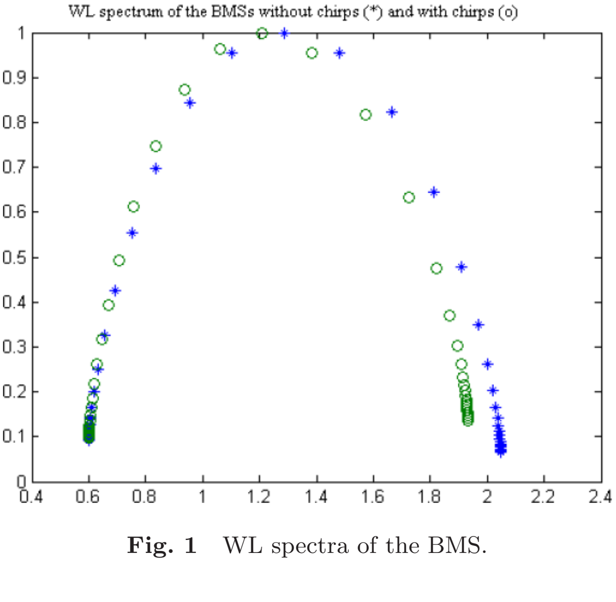

Fig. 1 WL spectra of the BMS.

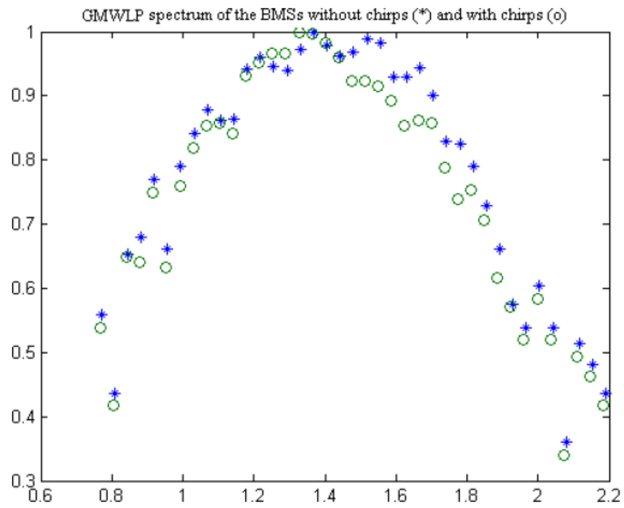

Fig. 2 GMWLP spectra of the BMS.

both methods the accuracy is very similar for all the considered values of a, so we show (Fig. 3) the graphics for $a = 0.75$ and for series of length 2^{13}.

A binomial multifractal series of length 2^{ω}, $x(s)$, with $s = 1, \ldots, 2^{\omega}$ can be obtained taking $x(s) = a^{\phi(k-1)}(1-a)^{\omega-\phi(k-1)}$ where $\phi(s)$ is the number of digits equal to 1 in the binary expansion of s [with $a \in (0.5, 1)$]. For instance, if $s = 27 = (11011)_2$ we have $\phi(s) = 4$.

For our samples we take symmetric series of length $2^{\omega+1}$, reflecting the BMS from right to left.

We also test the methods by adding to the former series sequences of eight chirps of the form: $C \cdot (1-t^2)^2|t|^{0.5}Ch(t)$ where $Ch(t)$ is a function whose frequency increases from 0 to 80 when $|t|$ goes from 1 to 0 and with a convex track in the time-frequency plane, and its graphic is (Fig. 4).

One can see (Fig. 1) that the spectra obtained with WL or GMWLP are almost unaltered by the chirps as was theoretically predicted, since the mentioned chirps have regularity C^{∞} in almost every point. Nevertheless, the wavelet leaders are those who guarantee these desirable results, since the majority of the current methods, based on coefficients wavelet or q-modulus of continuity, are notably altered by the presence of chirps.

Besides we test the methods against dyadic random wavelet cascades. We use an algorithm based on random cascades (RC) on wavelet dyadic trees. This algorithm builds a random multifractal series by specifying its discrete wavelet coefficients. We begin with a coefficient 1 for the lower scale and we obtain recursively the coefficients for the next scale by multiplying by $\pm 2^{-h}$ where h is a random variable with normal distribution (and with random sign) (see Kantelhardt *et al.*[22]). Taking directly this series the results are quite similar, but we also build series by concatenating or by inserting BMS or RC to obtain series with non-concave spectra.

5. A DISCUSSION ABOUT NON-CONCAVE SPECTRA

As we have already mentioned, the current methods use various hypotheses about the functions or signals with which they deal, hypotheses generally known as multifractal formalisms. But they always provide multifractal concave spectra (due to the fact that they use the Legendre transform, which needs

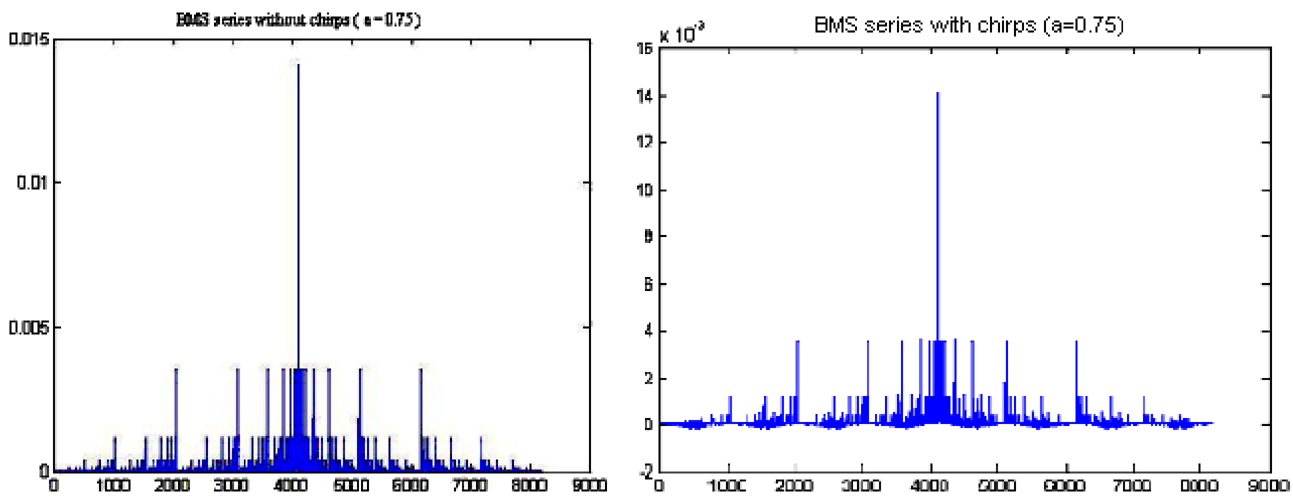

Fig. 3 BMS series.

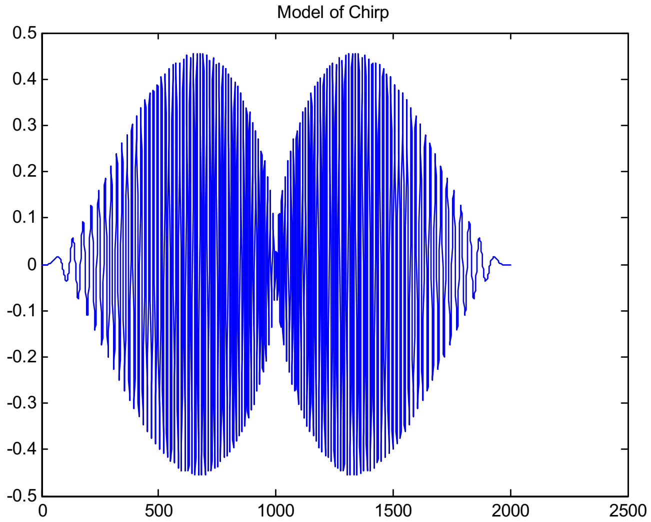

Fig. 4 Our model of Chirp.

a presumption of concavity). Notwithstanding, not all the signals or functions have got real concave spectra.

In many cases the real spectrum must be non-concave, and the methods that use multifractal formalisms only obtain, in the best case, its concave hull, or they are completely erroneous.

Let's build an explicit example of a function with non-concave spectrum:

(a) Let x be a BMS: $x(k)$, with a suitable value of a, such that their Hölder exponents lie in the interval $[0.4, 2.2]$ and its support is $[0, 1]$. This is always possible (see Kantelhardt *et al.*[22]).

(b) Afterwards we build another function whose spectrum will be the same as that of x but translated by two units; then their Hölder exponents lie in the interval: $[2.4, 4.2]$, and its support is included in $[0, 1]$. Is it possible to obtain such translation in the spectrum? Yes, by definition

$$d(\alpha) = \dim(E_\alpha) = \dim\{x/H_x(t) = \alpha\}$$

and it is sufficient to find a function

$z : \forall t \in [0, 1]$, $H_z(t) = H_x(t) + 2$. Now, $H(\cdot)$ is the Hölder exponent of a continuos functions if and only if it is the limit inferior of a secuence of continuous functions.[28,32] Then $H_x(t)$ and consequently $H_x(t) + 2$ are limit inferior of continuous functions.

Then there exists a continuous function z with $H_z(t) = H_x(t) + 2$, and then $\dim(E_{\alpha+2}(z)) = \dim(\{x \mid H_z(t) + 2 = \alpha + 2\} = \dim(\{x \mid H_x(t) = \alpha\} = \dim(E_\alpha)$. The demonstration of the theorem of Andersson's paper[32] suggests how to construct z by multiplying for 2^{2j} the values of the wavelet coefficients c_{jk} (and then the wavelet leaders change) obtaining $\lim_{j\to+\infty} \frac{\log_2 |L_{jk}(z)|}{-j} = H_Z(t_0)$ from the leaders corresponding to the neighborhoods of each $t_0 \in [0, 1]$.

(c) Taking $w(t) = z(t - 1)$ we have that the spectrum of w is the same as the spectrum of z, but the support of w is included in $[1, 2]$.

(d) Now we define

$$s(t) = \begin{cases} x(t) & \text{if } t \in [0, 1] \\ w(t) & \text{if } t \in [1, 2] \end{cases}$$

and as the level sets of the Hölder exponents of both intervals are disjoint (the dimension is not additive), we have that:

$$\begin{aligned} \text{if } \alpha \in [0.4, 2.2] \text{ then } & \dim(E_\alpha) \\ &= \dim(\{t \mid H_s(t) = \alpha\} \\ &= \dim(\{t \mid Hx(t) = \alpha\} \\ \text{if } \alpha \in [2.4, 4.2] \text{ then } & \dim(E_\alpha) \\ &= \dim(\{z \mid H_s(t) = \alpha\} \\ &= \dim(\{w \mid H_w(t) = \alpha\} \end{aligned}$$

and then the spectrum of s is non-concave (Fig. 5).

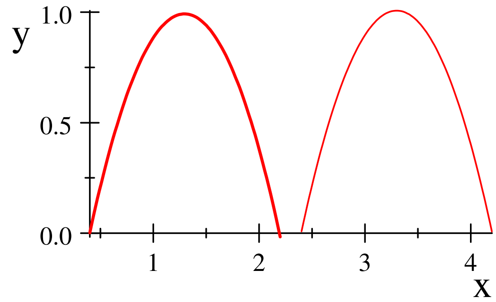

Fig. 5 Non-concave spectrum of the function $s(t)$.

After this theoretical example, let's return to the numerical estimations of the spectrum.

In Fig. 6 we show a series obtained by the following steps: a BMS is calculated as described above; then we generate a series with regularity at every point equal to that of the respective point of the BMS but increased in a certain fixed value ($\frac{4}{3}$ in the graphics but analogous results may be obtained with other values great enough); finally a new series is generated, mixing sections of the BMS and of the another series (the one obtained by increasing the regularity of the BMS).

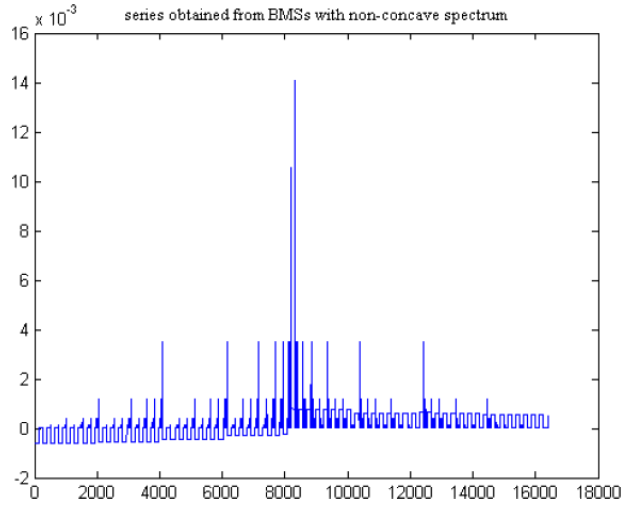

Fig. 6 Series obtained from BMS.

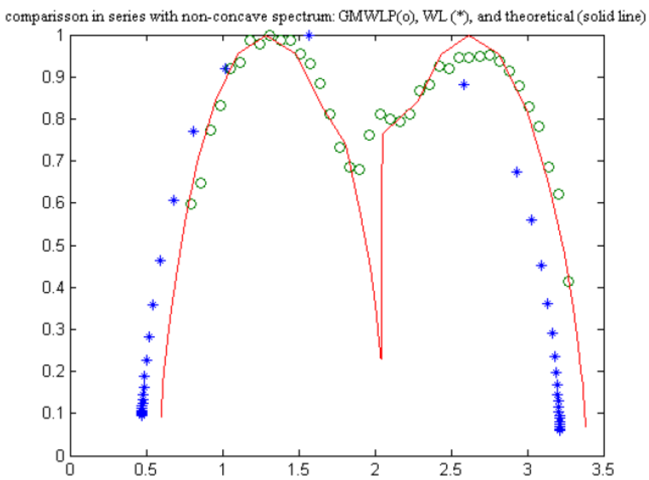

Fig. 7 Spectra of the series of Fig. 6.

As the dimensions are not altered, clearly the spectrum will be non-concave (the red solid line of Fig. 7). The method of WL, based on the Legendre transform, can only capture (approximately) the concave hull of the real spectrum. On the other hand, the GMWLP does not have the limitation of concavity, and the spectrum with GMWLP is a good approximation to the theoretical spectrum.

6. COMPARISON OF THE COMPUTATIONAL COMPLEXITY

None of the algorithms for computing the multifractal spectrum require a great amount of resources (neither the two we are taking in consideration nor any other known by us) so we will not pay attention to memory allocation and we will focus on studying the question about execution time.

Running any reasonably accurate implementation of the MF-DFA and of the WL will convince us that the latter method is faster than the former.

Let's analyze the number of operations (especially products) necessary for each method.

For the MF-DFA the critical steps occur when the local trends and the fluctuation function are calculated. The other steps only require $O(N)$ sums for obtaining the profile, no more than $O(N^2)$ indexing operations (the time required for this is despicable in comparison with the time required for products) and $O(N)$ products for taking log-log slopes and for performing the Legendre transform in order to finally computing the spectrum. But the $m-MFDFA$ requires, **for each window lengths** under consideration, at least

$$(m+2)^2 \cdot s \cdot 2N_s + (s+1) \cdot 2N_s + 2N_s + 2 \approx 2(m+2)^2 N$$

products (the first term is the greater one) to compute the least square fitting functions necessary for calculating the variances and for obtaining the fluctuation functions. We don't take in account a similar number of sums and differences. The number of window lengths to be considered vary from one implementation to another but always beginning with a size greater than $m+2$ but relatively small, 10 in our examples, and increasing up to a relatively great value of the order of N, in our examples $[\frac{N}{4}]$, and skipping arithmetically with increment 10 for our implementation. This give us an amount of about εN instances of the $2m^2N$ products mentioned above for each one of the values of q, where ε is an small constant, smaller than one: $\varepsilon N \approx \frac{1}{10}\frac{N}{4} = \frac{1}{40}N$ for us. So we have $2k(m+2)^2\varepsilon N^2$ products, where k is the number of values of q considered (21, between -10 and 10 for us). In the examples of this work we took the minimum order of m, that is $m = 1$, and so we have about $10N^2$ operations. Naturally it is possible to sacrifice some accuracy taking a greater skip from a window size to the next. But the factor $(m+2)^2$ cannot be reduced and it does not appear reasonable to take k much smaller: If we assume the usual shape ("almost parabolic") of the spectrum, the minimum sampling size would be about $k = 5$ if this method could guarantee exact results but this optimistic supposition is far from being true in real cases, especially if we must consider values of $|q| > 10$ (and this is not a rare case); so it does not seem to be useful to take $k < 10$. In any case the method is of order $O(N^2)$ and the constant cannot be taken very little if we hope to have a reasonable degree of accuracy. For the usual lower bounds for the sampling rate and the length of many series where we expect to have multifractal behavior it does not seem realistic to take a skip greater than 100 for window sizes (and perhaps the practical value must be quite smaller). So, the mentioned constant is at least approximately 1 and usually much larger in practice.

On the other hand, for the WL the most lengthy stage is the proper computing of the wavelet coefficients which is of order $O(N \log N)$ with a constant proportional to the number of nodes necessary to represent the orthonormal wavelet required. That is related to the number of null moments needed for to cancel the polynomials with degree up to the integer part of the maximal Hölder regularity of the signals. In the practice we rarely find Hölder regularity greater than 4 and then Daubechies wavelets of order 4 (like $db4$ or $sym4$) are enough. We use $sym4$ in the examples with $4.4 - 1 = 15$ nodes. If we would like to analyze specially smooth series we can take dbN or $symN$ with $4N - 1$ nodes but usually it is not necessary to take N greater than 8. WL also needs about $5N$ comparisons of double precision numbers, $O(N)$ sums or differences and $O(N)$ additional products, but this doesn't increases the order of the method.

Summarizing we have for the samples we have studied (mean length 2^{12}) that the WL method should be approximately $\frac{10}{15} \log_2(2^{12}) = 8$ times faster in agreement with what we have observed in the practice.

7. CONCLUSIONS

Besides the profitable considerations about the computational complexity mentioned in the previous section, it is necessary to emphasize the points that we describe now.

The tests previously realized suggest that the GMWLP is preferable for analyzing the series of EEG without losing some relevant information that is lost when applying methods that use the Legendre Transform, since the latter algorithms does not allow to visualize non-concave spectra.

With regard to this question, Durand,[33] gives examples of functions obtained from wavelet coefficients correlated by Markov chains in the torus having random not concave spectra, and he gives series with oscillating singularities almost everywhere of the torus.

Finally we remark that in signals artificially generated (like BMS or RC) the hypotheses are fulfilled and the methods provide similar spectra.

On the other hand, in signals from empirical sources the spectra calculated by the different methods differ considerably in certain signals.

Such difference is probable since, for example, in empirical EEG signals we cannot assure that necessarily they should correspond to an self-similar function or an approximately self-similar one.

Having a discreet signal for the analysis each method estimates the spectrum introducing hypothesis about the behavior of the unsampled values. In the case of methods that use the Legendre Transform the concavity of the scaling function is equivalent to a presumption of self-similarity (perhaps approximated) in every scales.

On the other hand the GMWLP assumes that we can estimate reasonably the distribution of the exponents Hölder from a fine enough histogram, and this implies that the function of Hölder exponents has not only bounded variation but also the diameter of the considered partition is sufficiently small with respect to this variation for estimation of the density with a reasonable precision.

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